Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 3 – calculus

Friday 18 November 2016 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 11]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{2x}{1+x^2}\right)y = x^2$, given that $y = 2$ when $x = 0$.

(a) Show that $1 + x^2$ is an integrating factor for this differential equation. [5]

(b) Hence solve this differential equation. Give the answer in the form $y = f(x)$. [6]

2. [Maximum mark: 18]

(a) By successive differentiation find the first four non-zero terms in the Maclaurin series for $f(x) = (x+1)\ln(1+x) - x$. [11]

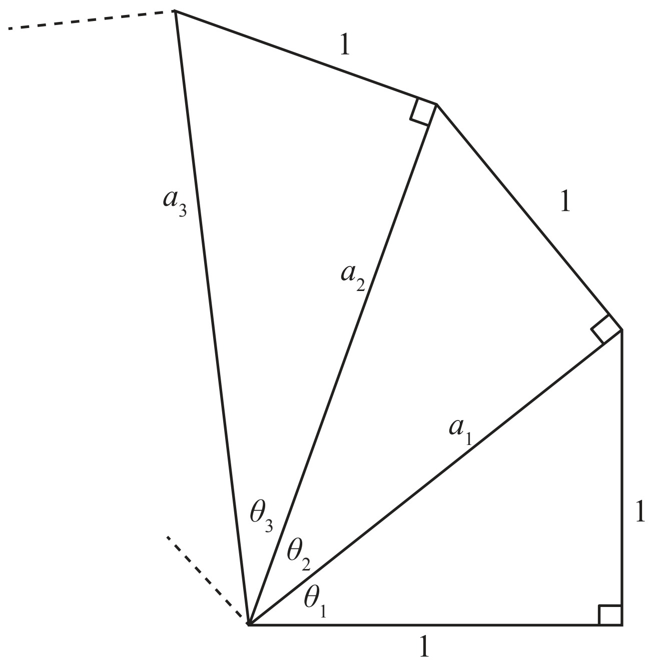
(b) Deduce that, for $n \geq 2$, the coefficient of x^n in this series is $(-1)^n \frac{1}{n(n-1)}$. [1]

(c) By applying the ratio test, find the radius of convergence for this Maclaurin series. [6]

3. [Maximum mark: 15]

(a) Using l'Hôpital's rule, find $\lim_{x \to \infty}\left(\dfrac{\arcsin\left(\dfrac{1}{\sqrt{(x+1)}}\right)}{\dfrac{1}{\sqrt{x}}}\right)$. [6]

Consider the infinite spiral of right angle triangles as shown in the following diagram.



The nth triangle in the spiral has central angle θ_n, hypotenuse of length a_n and opposite side of length 1, as shown in the diagram. The first right angle triangle is isosceles with the two equal sides being of length 1.

(b) (i) Find a_1 and a_2 and hence write down an expression for a_n.

(ii) Show that $\theta_n = \arcsin\dfrac{1}{\sqrt{(n+1)}}$. [3]

Consider the series $\sum_{n=1}^{\infty}\theta_n$.

(c) Using a suitable test, determine whether this series converges or diverges. [6]

Turn over

4. [Maximum mark: 16]

(a) State the mean value theorem for a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval $]a, b[$. [2]

Let $f(x)$ be a function whose first and second derivatives both exist on the closed interval $[0, h]$.

Let $g(x) = f(h) - f(x) - (h - x)f'(x) - \frac{(h - x)^2}{h^2}\left(f(h) - f(0) - hf'(0)\right)$.

(b) (i) Find $g(0)$.

(ii) Find $g(h)$.

(iii) Apply the mean value theorem to the function $g(x)$ on the closed interval $[0, h]$ to show that there exists c in the open interval $]0, h[$ such that $g'(c) = 0$.

(iv) Find $g'(x)$.

(v) Hence show that $-(h - c)f''(c) + \frac{2(h - c)}{h^2}\left(f(h) - f(0) - hf'(0)\right) = 0$.

(vi) Deduce that $f(h) = f(0) + hf'(0) + \frac{h^2}{2}f''(c)$. [9]

(c) Hence show that, for $h > 0$

$$1 - \cos(h) \leq \frac{h^2}{2}.$$ [5]